

Confidence Intervals

When gathering statistics using samples, there is always some uncertainty about how well the sample matches the overall population. Confidence intervals are a way to quantify that uncertainty.

We'll use an example as we explain the terms.

Suppose you want to estimate the mean mass of Canadian men. It is impractical to actually measure every man's mass, so instead you take a random sample of 128 men. In that sample you have the following:

$$\bar{x} = 175 \text{ pounds (mean)}$$

$$s = 25 \text{ pounds (standard deviation)}$$

$$n = 128 \text{ (sample size)}$$

Find the 95% confidence interval for the mean mass of Canadian men.

The components of a confidence interval

Parameter

This is the value we are measuring. In this example it is the mean mass of Canadian men.

Confidence level

This is a percentage, in this case 95%. This is the percentage of all samples whose confidence intervals are expected to include the actual population parameter.

This gives us a new value called the significance level:

$$\alpha = 1 - (\text{confidence level})$$

For example, if the confidence level is 95% = 0.95, the significance level is $1 - 0.95 = 0.05$.

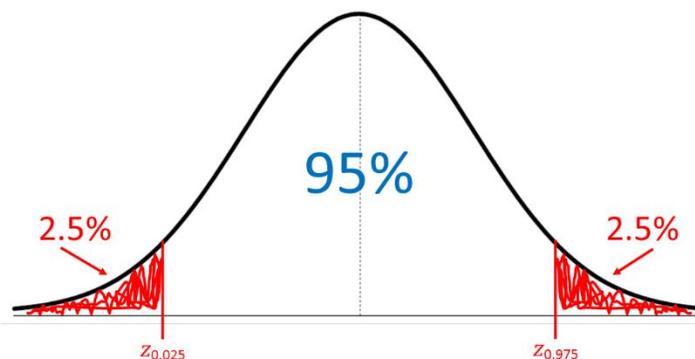
Margin of error

This is half of the size of the interval; it's the interval's range above and below the sample parameter. It's equal to

$$\text{Margin of error} = (\text{critical value}) \times (\text{standard error})$$

Critical value

Graphically, we want to find a z-score so that the confidence level is in the middle and the significance level is in the tails. Here's what it looks like for 95%:



The critical value is the positive z -score that does this for us. You can think of it as the z -score that leaves half of a significance level to the right, or

$$z_{1-\left(\frac{\alpha}{2}\right)}$$

For a 95% confidence interval we have $\alpha = 0.05$, so we're finding the critical value

$$\begin{aligned} z_{1-\left(\frac{0.05}{2}\right)} &= z_{1-0.025} \\ &= z_{0.975} \\ &= 1.96 \end{aligned}$$

z -scores don't have units, since they're a count of the number of standard deviations.

Standard error

The standard error is the standard deviation divided by the root of the sample size:

$$\begin{aligned} \frac{s}{\sqrt{n}} &= \frac{25}{\sqrt{128}} \\ &\doteq 2.21 \end{aligned}$$

This is sometimes denoted SE .

Margin of error, finally

Remember that

$$\begin{aligned} \text{Margin of error} &= (\text{critical value}) \times (\text{standard error}) \\ &= z_{1-\left(\frac{\alpha}{2}\right)} \frac{s}{\sqrt{n}} \\ ME &\doteq (1.96)(2.21) \\ &\doteq 4.33 \end{aligned}$$

This value is measured in the same units as the parameter, so in this case the margin of error is 4.33 pounds.

At last! Calculating a confidence interval

Choose the parameter (such as mean mass of Canadian men).

Choose a confidence level (such as 95%).

Calculate margin of error (4.33 pounds).

In general, the $1 - \alpha$ confidence interval is $\bar{x} \pm z_{1-\left(\frac{\alpha}{2}\right)} \frac{s}{\sqrt{n}}$, which is the sample mean plus-or-minus the margin of error.

The 95% confidence interval is 175 ± 4.33 pounds.

What does this mean?

This means that based on our sample of 128 men, we have a 95% confidence that the "true" population mean (the actual mean for ALL Canadian men) is between 170.67 and 179.33 pounds (that is, 175 ± 4.33 pounds).

Another way to think about this is that there is a 5% probability that we got a "bad" sample, and the "true" population mean is either less than 170.67 or more than 179.33.

How do confidence level and margin of error relate?

If you want to reduce your margin of error, you must also *reduce* your confidence level. If you're casting a smaller net, you'll be less certain of catching a fish.

If you want to have a higher confidence level, your margin of error must *increase*.

In the formula you can see this:

$$\bar{x} \pm z_{1-\left(\frac{\alpha}{2}\right)} \frac{s}{\sqrt{n}}$$

Everything depends on α . If α gets larger (because the confidence level decreases), the critical value $z_{1-\left(\frac{\alpha}{2}\right)}$ will decrease, and the margin of error will decrease. The opposite is also true.

How does sample size affect this?

The sample size, n , is part of the denominator for the standard error. As n increases, standard error decreases, and margin of error decreases.

An example

MTS deal rings hollow: 61% of Manitobans disapprove of sale to BCE

"Kurl said that as an online survey, it can't state a margin of error, but as a comparison, a sample of this size would carry a margin of error plus or minus 4.6 per cent 19 times out of 20."

(Source: <http://www.winnipegfreepress.com/business/mts-deal-rings-hollow-381499641.html>, retrieved on 2016-06-01)

They can't state a margin of error because they know there is significant bias in an online survey and they aren't able to account for it mathematically.

The "19 times out of 20" is the same as saying that this is a 95% confidence interval $\left(\frac{19}{20} = 0.95\right)$.

Since the parameter is the percentage of Manitobans who disapprove of the sale, we have a confidence interval of $61 \pm 4.6\%$ which is 56.4% to 65.6%.

Another way to say this is that there is a 5% chance that this sample isn't typical, and that the actual percentage of disapproving Manitobans is outside of that interval.